NEW EXAMPLES OF WILLMORE SUBMANIFOLDS IN THE UNIT SPHERE VIA ISOPARAMETRIC FUNCTIONS

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Dedicated to Professor Chiakuei Peng on his 70-th birthday.

ABSTRACT. An isometric immersion $x : M^n \to S^{n+p}$ is called Willmore if it is an extremal submanifold of the Willmore functional: $W(x) = \int_{M^n}(S - nH^2)^{\frac{n}{2}} dv$, where S is the norm square of the second fundamental form and H is the mean curvature. Examples of Willmore submanifolds in the unit sphere are scarce in the literature. The present paper gives a series of new examples of Willmore submanifolds in the unit sphere via isoparametric functions of FKM-type.

1. Introduction

Let M be an n-dimensional compact submanifold immersed in an $(n+p)$-dimensional unit sphere S^{n+p}. Denote by h the second fundamental form of M, S the norm square of h, $\vec{\mathbf{H}}$ the mean curvature vector and H the mean curvature of M, respectively. Based on the following range of indices:

$$1 \le i, j, k \le n; \quad n+1 \le \alpha, \beta, \gamma \le n+p; \quad 1 \le A, B, C \le n+p,$$

we let $\{e_A\}$ be a field of local orthonormal basis for TS^{n+p} such that when restricted to M^n, $\{e_i\}$ is a field of local orthonormal basis for TM and $\{e_\alpha\}$ is a field of local orthonormal basis for the normal bundle $T^\perp M$. In this way, h has components h^α_{ij}, and we immediately get the following expressions:

$$S = \sum_{\alpha,i,j}(h^\alpha_{ij})^2, \quad \vec{\mathbf{H}} = \sum_\alpha H^\alpha e_\alpha, \quad H^\alpha = \frac{1}{n}\sum_i h^\alpha_{ii}, \quad H = |\vec{\mathbf{H}}|.$$

M^n is called a *Willmore submanifold* in S^{n+p} if it is an extremal submanifold of the Willmore functional, which is conformal invariant (*cf.* [Wan1]):

$$W(x) = \int_{M^n}(S - nH^2)^{\frac{n}{2}} dv.$$

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Defining a non-negative function ρ^2 on M:

$$\rho^2 = S - nH^2,$$

we recall an equivalent condition for M to be Willmore:

Theorem ([GLW], [PW]) *M^n is a Willmore submanifold in S^{n+p} if and only if for any α with $n+1 \leq \alpha \leq n+p$,*

$$\begin{aligned} -\rho^{n-2}\sum_{i,j}\Big(R_{ij}-(n-1)\sum_{\beta}H^{\beta}h^{\beta}_{ij}\Big)\Big(h^{\alpha}_{ij}-H^{\alpha}\delta_{ij}\Big) \\ +(n-1)\Delta(\rho^{n-2}H^{\alpha})-\sum_{i,j}(\rho^{n-2})_{,ij}(nH^{\alpha}\delta_{ij}-h^{\alpha}_{ij})=0, \end{aligned} \tag{1}$$

where R_{ij} is the Ricci tensor.

Remark 1.1. When $n = 2$, the formula (1) reduces to the following well known criterion of Willmore surfaces:

$$\Delta^{\perp}H^{\alpha}+\sum_{\beta,i,j}H^{\beta}h^{\alpha}_{ij}h^{\beta}_{ij}-2|\vec{\mathbf{H}}|^2H^{\alpha}=0, \qquad 3\leq\alpha\leq 2+p. \tag{2}$$

Clearly, every minimal surface in S^{2+p} is automatically Willmore; in other words, Willmore surfaces are a generalization of minimal surfaces in a sphere. □

It follows immediately from the theorem above that all n-dimensional Einstein manifolds minimally immersed in the unit sphere are Willmore submanifolds. However, as a matter of fact, there do exist examples of Willmore hypersurfaces in the unit sphere, which are neither minimal nor Einstein. For instance, Cartan's minimal isoparametric hypersurface is Willmore but not Einstein, while one certain hypersurface in each of Nomizu's isoparametric families is Willmore, but neither minimal nor Einstein. In addition, [Li] characterized all isoparametric Willmore hypersurfaces in the unit sphere.

In this paper, by taking advantage of isoparametric functions of FKM-type, we give a series of new examples of Willmore submanifolds M^n in the unit sphere S^{n+p}, which are minimal but in general not Einstein (for details, see the concluding remark).

As is well known, a hypersurface M^n in the unit sphere S^{n+1} is *isoparametric* if it is a level hypersurface of some locally defined isoparametric function f on S^{n+1}, that is, a non-constant smooth function $f : S^{n+1} \to \mathbb{R}$ (*cf.* [Wan2]) satisfying:

$$\begin{cases} |\nabla f|^2 = b(f) \\ \triangle f \ = a(f) \end{cases} \tag{3}$$

where ∇f and $\triangle f$ is the gradient and Laplacian of f, respectively, b is a smooth function on $\mathbb{R}$, and a is a continuous function on $\mathbb{R}$.

Élie Cartan (*cf.* [Car1, Car2, Car3, Car4]) pointed out that the level hypersurfaces $M_t := f^{-1}(t)$ corresponding to regular values t of f are parallel (which are called

isoparametric hypersurfaces) and have constant principal curvatures. It is well known that the focal submanifolds $M_+ := f^{-1}(+1)$ and $M_- := f^{-1}(-1)$ are minimal submanifolds in S^{n+1} (*cf.* [CR]). In fact, as asserted by Ge and Tang ([GT]), this result still holds for focal submanifolds of isoparametric hypersurfaces in any Riemannian manifold.

We now recall the construction of isoparametric functions of FKM-type. For a symmetric Clifford system $\{P_0, \cdots, P_m\}$ on $\mathbb{R}^{2l}$, *i.e.* P_i's are symmetric matrices satisfying $P_iP_j + P_jP_i = 2\delta_{ij}I_{2l}$, Ferus, Karcher and Münzner ([FKM]) constructed a polynomial F on $\mathbb{R}^{2l}$:

$$\begin{aligned} F: \quad & \mathbb{R}^{2l} \to \mathbb{R} \\ F(x) &= |x|^4 - 2\sum_{i=0}^{m} \langle P_i x, x\rangle^2. \end{aligned} \tag{4}$$

It is not difficult to verify that $f = F|_{S^{2l-1}}$ satisfies (*cf.*[Ce], [GTY]):

$$\begin{cases} |\nabla f|^2 = 16(1-f^2) \\ \triangle f \ = 8(m_2 - m_1) - 4(2l+2)f. \end{cases} \tag{5}$$

where $m_1 = m$, $m_2 = l - m - 1$ are the differences of the dimensions of M_+ and M_- compared with that of the isoparametric hypersurface, respectively, ∇f and $\triangle f$ are the gradient and Laplacian of f on S^{2l-1}, respectively. Thus by definition, f is an *isoparametric function* on S^{2l-1}, which is called the *FKM-type isoparametric polynomial* on S^{2l-1}. Correspondingly, the focal submanifolds $M_+ := f^{-1}(+1)$ can be expressed as an algebraic set:

$$M_+ = \{x \in S^{2l-1} \mid \langle P_0x, x\rangle = ... = \langle P_m x, x\rangle = 0\ \},$$

which will be exactly our new examples of Willmore submanifolds in S^{2l-1}.

The main result of the present paper is stated as follows:

Theorem 1.1. *The focal submanifolds M_+ of the FKM-type isoparametric polynomials on S^{2l-1} are Willmore submanifolds in S^{2l-1}.*

2. Proof of Theorem 1.1

We start with a description of the normal space of the focal submanifold M_+ following [FKM]. Since $M_+ = \{\ x \in S^{2l-1} \mid \langle P_0x, x\rangle = ... = \langle P_m x, x\rangle = 0\ \}$, it follows without much difficulty that $dim\ M_+ = 2l - 1 - (m+1) = 2l - m - 2$, and as pointed out by [FKM], the normal space at $x \in M_+$ is

$$T_x^{\perp} M_+ = \{\ \mathcal{P}x \mid \mathcal{P} \in \mathbb{R}\Sigma(P_0, ..., P_m)\ \}, \tag{6}$$

where $\Sigma(P_0, ..., P_m)$ is the unit sphere in $Span\{P_0, ..., P_m\}$, which is called *the Clifford sphere* determined by the system $\{P_0, ..., P_m\}$.

For a normal vector $\xi_\alpha = P_\alpha x$, $\alpha = 0, ..., m$, let $\langle A_\alpha X, Y\rangle = \langle h(X,Y), \xi_\alpha\rangle$, $\forall X, Y \in T_xM_+$, where A_α is the shape operator corresponding to ξ_α; in other words, we have $h(X,Y) = \sum_\alpha \langle A_\alpha X, Y\rangle \xi_\alpha$.

Next, from Gauss equation, we derive that

$$K(X,Y) = 1 + \sum_{\alpha=0}^{m} \Big\{\langle A_\alpha X, X\rangle\langle A_\alpha Y, Y\rangle - \langle A_\alpha X, Y\rangle^2\Big\}, \tag{7}$$

where K is the sectional curvature in M_+. Furthermore, in virtue of the fact $A_\alpha X = -(P_\alpha X)^T$ (*cf.* [FKM]), the tangential component of $-P_\alpha X$, the formula (7) becomes:

$$K(X,Y) = 1 + \sum_{\alpha=0}^{m} \Big\{\langle P_\alpha X, X\rangle\langle P_\alpha Y, Y\rangle - \langle P_\alpha X, Y\rangle^2\Big\} \tag{8}$$

Let $\{X = e_1, e_2, ..., e_{2l-m-2}\}$ be an orthonormal basis for T_xM_+, and $\{P_0x, ..., P_mx\}$ an orthonormal basis for $T_x^\perp M_+$. Then the Ricci curvature for X is

$$\begin{aligned} Ricci(X) &= \sum_{i=2}^{2l-m-2} K(X, e_i) \\ &= 2l-m-3 + \sum_{i=2}^{2l-m-2}\sum_{\alpha=0}^{m}\Big\{\langle P_\alpha X, X\rangle\langle P_\alpha e_i, e_i\rangle - \langle P_\alpha X, e_i\rangle^2\Big\} \\ &= 2l-m-3 - \sum_{\alpha=0}^{m}\Big\{\langle P_\alpha X, X\rangle^2 + \sum_{i=2}^{2l-m-2}\langle P_\alpha X, e_i\rangle^2\Big\} \\ &= 2(l-m-2) + 2\sum_{\alpha,\beta=0,\alpha<\beta}^{m}\langle X, P_\alpha P_\beta x\rangle^2, \end{aligned} \tag{9}$$

where the third equality is derived from the fact that for any $\alpha \in \{0, ..., m\}$, P_α is trace free, and

$$\begin{aligned} trace\ P_\alpha &= \langle P_\alpha X, X\rangle + \sum_{i=2}^{2l-m-2}\langle P_\alpha e_i, e_i\rangle + \sum_{\beta=0}^{m}\langle P_\alpha P_\beta x, P_\beta x\rangle + \langle P_\alpha x, x\rangle \\ &= \langle P_\alpha X, X\rangle + \sum_{i=2}^{2l-m-2}\langle P_\alpha e_i, e_i\rangle, \end{aligned} \tag{10}$$

and the fourth equality is obtained by the relation that for any $\alpha \in \{0, ..., m\}$,

$$\begin{aligned} 1 &= |P_\alpha X|^2 \\ &= \langle P_\alpha X, X\rangle^2 + \sum_{i=2}^{2l-m-2}\langle P_\alpha X, e_i\rangle^2 + \sum_{\beta=0}^{m}\langle P_\alpha X, P_\beta x\rangle^2 + \langle P_\alpha X, x\rangle^2 \\ &= \langle P_\alpha X, X\rangle^2 + \sum_{i=2}^{2l-m-2}\langle P_\alpha X, e_i\rangle^2 + \sum_{\beta=0}^{m}\langle X, P_\alpha P_\beta x\rangle^2. \end{aligned} \tag{11}$$

Moreover, the the norm square of the second fundamental form of M_+ can be calculated immediately

$$
\begin{aligned}
(12) \qquad S &= \sum_{i,j=1}^{2l-m-2} \sum_{\alpha=0}^{m} \langle A_\alpha e_i, e_j\rangle^2 \\
&= \sum_{i,j=1}^{2l-m-2} \sum_{\alpha=0}^{m} \langle P_\alpha e_i, e_j\rangle^2 \\
&= \sum_{i=1}^{2l-m-2} \sum_{\alpha=0}^{m} \Big\{1 - \sum_{\beta=0}^{m} \langle P_\alpha e_i, P_\beta x\rangle^2\Big\} \\
&= (2l-m-2)(m+1) - 2 \sum_{\alpha,\beta=0,\alpha<\beta}^{m} |P_\alpha P_\beta x|^2 \\
&= 2(l-m-1)(m+1).
\end{aligned}
$$

Remark 2.1. In fact, for every isoparametric hypersurface with four distinct principal curvatures in the unit sphere, the norm square S of the second fundamental forms of both focal submanifolds are constant. The proof is a consequence of a result of Münzner (cf. [CR] page 248).

As a direct result of (12), $\rho^2 = S - nH^2 = S$ is a constant on M_+. Together with the minimality of M_+, *i.e.* $H^\alpha = 0$ for any $0 \le \alpha \le m$, the criterion (1) for Willmore reduces to

$$
(13) \qquad for\ any\ 0 \le \alpha \le m, \quad \sum_{i,j=1}^{2l-m-2} R_{ij} h_{ij}^\alpha = 0.
$$

In order to prove that (13) holds for M_+, we first note that for any $\mathcal{P} \in \Sigma(P_0, ..., P_m)$, the eigenvalues of $\mathcal{P}$ must be ± 1, since $\mathcal{P}^2 = I$. Observe that since $trace\ \mathcal{P} = 0$, E_+ and E_-, the eigenspaces of $\mathcal{P}$ for the eigenvalues $+1$ and -1, have the same dimension l, so that $\mathbb{R}^{2l} = E_+ \oplus E_-$. Then we can compute the principal curvatures of the shape operator A_ξ, with respect to a unit normal $\xi = \mathcal{P}x$ as follows.

Lemma ([Ce], [FKM]) *Let x be a point in the focal submanifold M_+, and let $\xi = \mathcal{P}x$ be a unit normal vector to M_+ at x, where $\mathcal{P} \in \Sigma(P_0, ..., P_m) := \Sigma$. Then the shape operator A_ξ has principal curvatures 0, 1, -1 with corresponding principal spaces $T_0(\xi)$, $T_1(\xi)$ and $T_{-1}(\xi)$ as follows:*

$$
\begin{aligned}
(14) \qquad T_0(\xi) &= \{\ QPx \mid Q \in \Sigma,\ \langle Q, P\rangle = 0\ \}, \\
T_1(\xi) &= \{\ X \in E_- \mid X \cdot Qx = 0,\ \forall Q \in \Sigma\ \} = E_- \cap T_x M_+, \\
T_{-1}(\xi) &= \{\ X \in E_+ \mid X \cdot Qx = 0,\ \forall Q \in \Sigma\ \} = E_+ \cap T_x M_+.
\end{aligned}
$$

Furthermore,

$$dimT_0(\xi) = m, \quad dimT_1(\xi) = dimT_{-1}(\xi) = l - m - 1.$$

In this way, T_xM_+ can be decomposed as a direct sum $T_xM_+ = T_0(\xi) \oplus T_1(\xi) \oplus T_{-1}(\xi)$. Let's choose an orthonormal basis $\{u_1, ..., u_m, v_1, ..., v_{l-m-1}, w_1, ..., w_{l-m-1}\}$ on T_xM_+ with $u_1, ..., u_m \in T_0(\xi)$, $v_1, ..., v_{l-m-1} \in T_1(\xi)$ and $w_1, ..., w_{l-m-1} \in T_{-1}(\xi)$. Then it is easy to see

$$\sum_{i,j=1}^{2l-m-2} R_{ij}h^{\alpha}_{ij} = \sum_{i=1}^{l-m-1} \{Ricci(v_i) - Ricci(w_i)\}.$$

Thus it follows from (13) that

$$M_+ \ is \ Willmore \ \Leftrightarrow \sum_{i=1}^{l-m-1} Ricci(v_i) = \sum_{i=1}^{l-m-1} Ricci(w_i). \tag{15}$$

However, by (9), we have

$$\begin{aligned} & \sum_{i=1}^{l-m-1} Ricci(v_i) = \sum_{i=1}^{l-m-1} Ricci(w_i) \\ \Leftrightarrow \ & \sum_{i=1}^{l-m-1} \sum_{\alpha,\beta=0}^{m} \langle P_\alpha v_i, P_\beta x\rangle^2 = \sum_{i=1}^{l-m-1} \sum_{\alpha,\beta=0}^{m} \langle P_\alpha w_i, P_\beta x\rangle^2 \\ \Leftrightarrow \ & \sum_{\alpha,\beta=0,\alpha\neq\beta}^{m} |(P_\alpha P_\beta x)^{T_1}|^2 = \sum_{\alpha,\beta=0,\alpha\neq\beta}^{m} |(P_\alpha P_\beta x)^{T_{-1}}|^2 \end{aligned} \tag{16}$$

Finally, we notice that when $m = 1$, the rightmost side of (16) automatically holds. In order to complete the proof of the theorem, we need the following observation.

Given any $\mathcal{P} \in \Sigma(P_0, ..., P_m)$, we can assume $\mathcal{P} = P_0$ without loss of generality (choose a suitable orthogonal transformation on $\mathbb{R}\Sigma(P_0, ..., P_m)$ preserving the geometric equivalence class). Under this assumption, we get a consequence that for $X \in T_1(\xi)$, $P_0X = -X$, and for $Y \in T_{-1}(\xi)$, $P_0Y = Y$. In fact, by $(P_0X)^T = -A_\xi X = -X$, we can decompose P_0X as $P_0X = -X + (P_0X)^N$, where $(P_0X)^N$ is the normal part of P_0X. Then from the identities $|P_0X| = |X| = 1$, we derive that $P_0X = -X$. The other proof with respect to Y is analogous.

Now with this preparation, it suffices to prove the rightmost side of (16) in the following two cases.

Case A: $m = 2$. In this case, P_0P_1x, $P_0P_2x \in T_0(\xi)$, we just need to deal with P_1P_2x. Since $\langle P_1P_2x, P_\alpha x\rangle = 0$ for any $\alpha = 0, 1, 2$, and $\langle P_1P_2x, x\rangle = 0$, we know $P_1P_2x \in T_xM_+$.

Decompose P_1P_2x as $P_1P_2x = U+V+W \in T_0(\xi)\oplus T_1(\xi)\oplus T_{-1}(\xi)$. In virtue of the observation, we have $P_0P_1P_2x = P_0U - V + W = -V + W$. In fact, according to (14), we can write $U = Q_0P_0x$ with $Q_0 \in \Sigma$, so that $P_0U = -Q_0x \in T_x^{\perp}M_+$. On the other hand, since $\langle P_0P_1P_2x, P_\alpha x\rangle = 0$ for any $\alpha = 0, 1, 2$, we know that $P_0P_1P_2x \in T_xM_+$. Thus $P_0U = 0$.

From $\langle P_1P_2x, P_0P_1P_2x\rangle = 0$, it follows directly that $\langle -V + W, U + V + W\rangle = -|V|^2 + |W|^2 = 0$, or equivalently, $|(P_1P_2x)^{T_1}| = |(P_1P_2x)^{T_{-1}}|$.

Case B: $m > 2$. In this case, $P_0P_\alpha x \in T_0(\xi)$ for any $\alpha = 1, ..., m$, so we just need to deal with $P_\alpha P_\beta x$ for $\alpha, \beta > 0$. Observing that for $\alpha \neq \beta$, $\langle P_\alpha P_\beta x, P_\gamma x\rangle = 0$ for any $\gamma = 0, ..., m$, and $\langle P_\alpha P_\beta x, x\rangle = 0$, we have $P_\alpha P_\beta x \in T_xM_+$.

Decomposing $P_\alpha P_\beta x$ as $P_\alpha P_\beta x = U + V + W \in T_0(\xi) \oplus T_1(\xi) \oplus T_{-1}(\xi)$, we get $P_0P_\alpha P_\beta x = P_0U - V + W$, and

$$\left\{\begin{array}{l} P_\alpha P_\beta x + P_0P_\alpha P_\beta x = U + P_0U + 2W \\ P_\alpha P_\beta x - P_0P_\alpha P_\beta x = U - P_0U + 2V. \end{array}\right. \tag{17}$$

Taking norm square on both sides of the two equalities above, since $\langle P_0P_\alpha P_\beta x, P_\alpha P_\beta x\rangle = 0$ for any $\alpha, \beta > 0$, we arrive at

$$\left\{\begin{array}{l} 2 = |U|^2 + |P_0U|^2 + 4|W|^2 \\ 2 = |U|^2 + |P_0U|^2 + 4|V|^2, \end{array}\right. \tag{18}$$

which implies $|V|^2 = |W|^2$, *i.e.* $|(P_\alpha P_\beta x)^{T_1}| = |(P_\alpha P_\beta x)^{T_{-1}}|$.

The proof of our main theorem is now complete!

Concluding Remark. In general, the focal submanifold M_+ of the FKM-type isoparametric function is not Einstein. As is well known, for each positive integer k, there exists one Clifford system $\{P_0, ..., P_m\}$ on $\mathbb{R}^{2l}$ with $l = k\delta(m)$, where $\delta(m)$ has the following values:

m	1	2	3	4	5	6	7	8	$\cdots$ $m{+}8$
$\delta(m)$	1	2	4	4	8	8	8	8	$16\delta(m)$

Since each focal manifold of the FKM-type isoparametric function has reduced dimension compared with the isoparametric hypersurface, we have $m_1 = m \geq 1$ and $m_2 = l - m - 1 \geq 1$. On these two conditions, with a fundamental but complicated argument (the details are omitted), we observe that if $m = 1, 2, 3$ or $m > 10$, then $4l > m^2 + 3m + 4$, which implies that

$$dim\ M_+ > \frac{1}{2}m(m+1) \geq dim\ Span\{P_\alpha P_\beta x \mid \alpha, \beta = 0, 1, ..., m, \alpha < \beta\}.$$

At last, combining with (9), we can conclude that the Ricci curvature of M_+ is not constant in these cases, namely, M_+ is not an Einstein manifold !

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